$$\langle n l m_l m_s | \delta H_{SB} | n l m_l m_s \rangle$$

$$= \langle n l m_l m_s | \lambda (\vec{L} \cdot \vec{S}) | n l m_l m_s \rangle$$

$$= \lambda \langle n l m_l m_s | L_x S_x + L_y S_y + L_z S_z | n l m_l m_s \rangle$$

$$\langle n' l' m_l' m_s' | L_x S_x | n l m_l m_s \rangle$$

$$= \langle n' l' m_l' m_s' | S_x \frac{L_+ + L_-}{2} | n l m_l m_s \rangle$$

$$= \frac{1}{2} \langle n' l' m_l' m_s' | S_x L_+ | n l m_l m_s \rangle + \frac{1}{2} \langle n' l' m_l' m_s' | S_x L_- | n l m_l m_s \rangle$$

$$= \frac{1}{2} \langle n' l' m_l' m_s' | S_x \hbar \sqrt{l(l+1) - m_l(m_l+1)} | n l (m_l+1) m_s \rangle$$

$$+ \frac{1}{2} \langle n' l' m_l' m_s' | S_x \hbar \sqrt{l(l+1) - m_l(m_l-1)} | n l (m_l-1) m_s \rangle$$

$$= \frac{\hbar}{2} \sqrt{l(l+1) - m_l(m_l+1)} \langle n' l' m_l' m_s' | \frac{S_+ + S_-}{2} | n l (m_l+1) m_s \rangle$$

$$+ \frac{\hbar}{2} \sqrt{l(l+1) - m_l(m_l-1)} \langle n' l' m_l' m_s' | \frac{S_+ + S_-}{2} | n l (m_l-1) m_s \rangle$$

$$\langle n' l' m_l' m_s' | S_+ + S_- | n l (m_l+1) m_s \rangle$$

$$= \langle n' l' m_l' m_s' | \hbar \sqrt{s(s+1) - m_s(m_s+1)} | n l (m_l+1) (m_s+1) \rangle$$

$$+ \langle n' l' m_l' m_s' | \hbar \sqrt{s(s+1) - m_s(m_s-1)} | n l (m_l+1) (m_s-1) \rangle$$

$$= \hbar \sqrt{s(s+1) - m_s(m_s+1)} \, \delta_{nn'} \delta_{ll'} \delta_{m_l' (m_l+1)}$$

$$= \hbar\sqrt{s(s+1) - m_s(m_s+1)}\ \delta_{nn'}\delta_{ll'}\delta_{(m_l+1)m_l'}\delta_{(m_s+1)m_s'}$$

$$+ \hbar\sqrt{s(s+1) - m_s(m_s-1)}\ \delta_{nn'}\delta_{ll'}\delta_{(m_l+1)m_l'}\delta_{(m_s-1)m_s'}$$

$$\langle n'l'm_l'm_s' \mid S_+ + S_- \mid nl\,(m_l-1)\,m_s\rangle$$

$$= \hbar\sqrt{s(s+1) - m_s(m_s+1)}\ \underbrace{\langle n'l'm_l'm_s' \mid nl\,(m_l-1)(m_s+1)\rangle}_{\delta_{nn'}\delta_{ll'}\delta_{(m_l-1)m_l'}\delta_{(m_s+1)m_s'}}$$

$$+ \hbar\sqrt{s(s+1) - m_s(m_s-1)}\ \underbrace{\langle n'l'm_l'm_s' \mid nl\,(m_l-1)(m_s-1)\rangle}_{\delta_{nn'}\delta_{ll'}\delta_{(m_l-1)m_l'}\delta_{(m_s-1)m_s'}}$$

$$\Longrightarrow$$

$$\langle n'l'm_l'm_s' \mid L_x S_x \mid nl\,m_l m_s\rangle$$

$$= \frac{\hbar^2}{4}\sqrt{l(l+1) - m_l(m_l+1)}\left(\sqrt{s(s+1) - m_s(m_s+1)}\ \delta_{nn'}\delta_{ll'}\delta_{(m_l+1)m_l'}\sqrt{\delta_{(m_s+1)m_s'}}\right.$$

$$\left. + \sqrt{s(s+1) - m_s(m_s-1)}\ \delta_{nn'}\delta_{ll'}\delta_{(m_l+1)m_l'}\delta_{(m_s-1)m_s'}\right)$$

$$+ \frac{\hbar^2}{4}\sqrt{l(l+1) - m_l(m_l-1)}\left(\sqrt{s(s+1) - m_s(m_s+1)}\ \delta_{nn'}\delta_{ll'}\delta_{(m_l-1)m_l'}\delta_{(m_s+1)m_s'}\right.$$

$$\left. + \sqrt{s(s+1) - m_s(m_s-1)}\ \delta_{nn'}\delta_{ll'}\delta_{(m_l-1)m_l'}\delta_{(m_s-1)m_s'}\right)$$