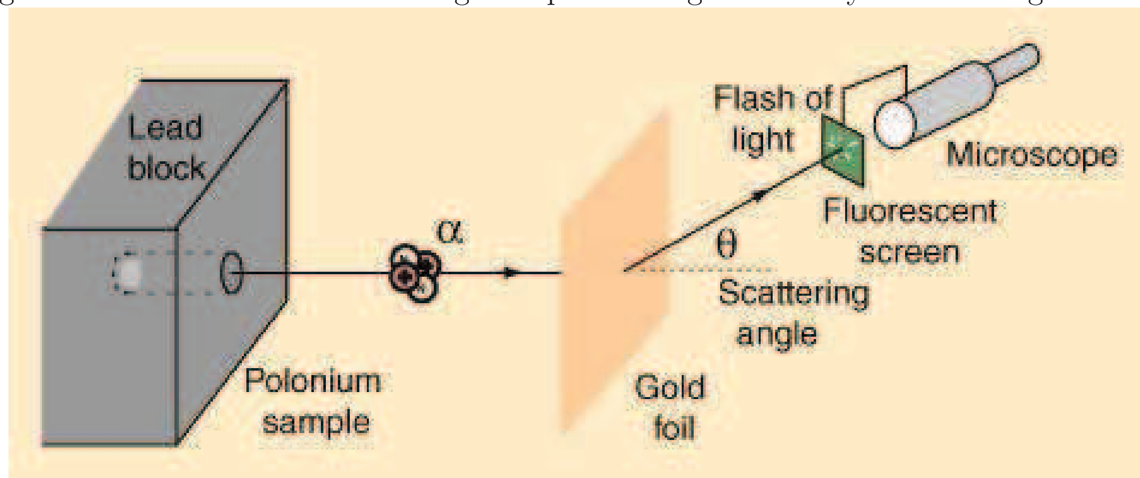
Chapter 2

Rutherford Scattering

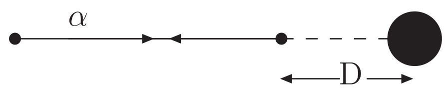
Let us start from the one of the first steps which was done towards understanding the deepest structure of matter. In 1911, Rutherford discovered the nucleus by analysing the data of Geiger and Marsden on the scattering of α-particles against a very thin foil of gold.



The data were explained by making the following assumptions.

- The atom contains a nucleus of charge Ze, where Z is the atomic number of the atom (i.e. the number of electrons in the neutral atom),
- The nucleus can be treated as a point particle,
- The nucleus is sufficently massive compared with the mass of the incident α-particle that the nuclear recoil may be neglected,
- That the laws of classical mechanics and electromagnetism can be applied and that no other forces are present,
- That the collision is elastic.

If the collision between the incident particle whose kinetic energy is T and electric charge ze ($z = 2$ for an α-particle), and the nucleus were head on,



the distance of closest approach D is obtained by equating the initial kinetic energy to the Coulomb energy at closest approach, i.e.

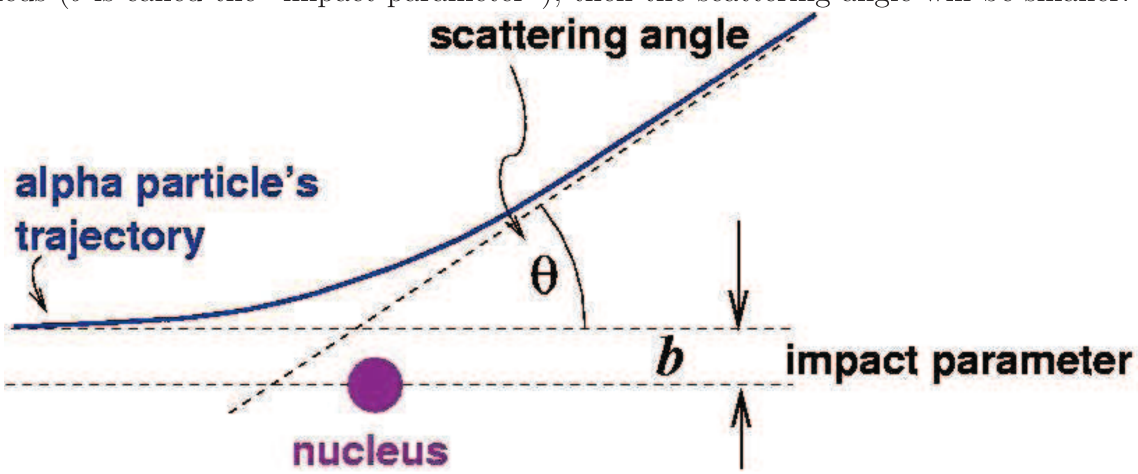
$$T = \frac{z\,Z\,e^2}{4\pi\epsilon_0 D},$$

or

$$D = \frac{z\,Z\,e^2}{4\pi\epsilon_0 T}$$

at which point the α-particle would reverse direction, i.e. the scattering angle θ would equal π.

On the other hand, if the line of incidence of the α-particle is a distance b, from the nucleus (b is called the "impact parameter"), then the scattering angle will be smaller.

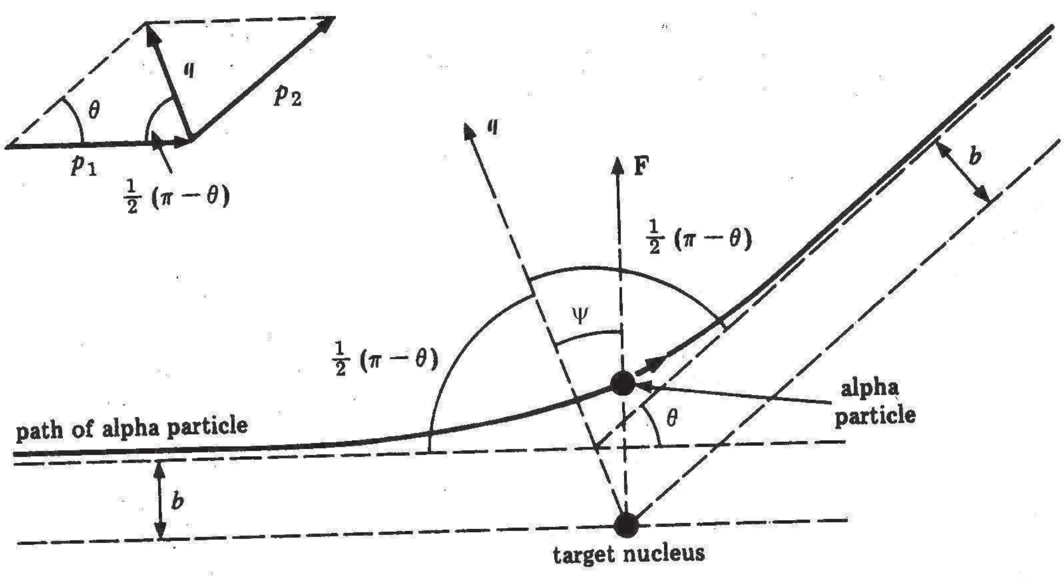


2.1 Relation between scattering angle and an impact parameter

The relation between b and θ is given by

$$\tan\left(\frac{\theta}{2}\right) = \frac{D}{2b} \tag{2.1.1}$$

This relation is derived using Newton's Second Law of Motion, Coulomb's law for the force between the α-particle and and nucleus, and conservation of angular momentum. The derivation is given in this section. Here we note that $\theta = \pi$ when $b = 0$ as stated above and that as b increases the α-particle 'glances' the nucleus so that the scattering angle decreases.



Geometrical relationships in Rutherford scattering.

The initial and final momenta, p_1, p_2 are equal in magnitude (p) (recall, that, elastic scattering is assumed), so that together with the momentum change $\mathbf{q}$ they form an isosceles triangle with angle θ between the initial and final momenta, as shown above.

Using the sine rule we have

$$\frac{q}{p} = \frac{\sin\theta}{\sin\left(\frac{1}{2}(\pi-\theta)\right)} \;=\; 2\sin\left(\frac{\theta}{2}\right). \tag{2.1.2}$$

The direction of the vector $\mathbf{q}$ is along the line joining the nucleus to the point of closest approach of the α-particle.

We assume that the nucleus is much heavier than the α-particle so we can neglect its recoil. We also neglect any relativistic effects.

The position of the α-particle is given in terms of two-dimensional polar coordinates r, ψ with the nucleus as the origin and $\psi = 0$ chosen to be the point of closest approach.

By Newton's second law, the rate of change of momentum in the direction of $\mathbf{q}$ is the component of the force acting on the α-particle due to the electric charge of the nucleus. By Coulomb's law the magnitude of the force is

$$F \;=\; \frac{zZe^2}{4\pi\epsilon_0 r^2},$$

where $Z\,e$ is the electric charge of the nucleus, and $z\,e$ is the electric charge of the incident particle (for an α-particle $z = 2$). Using $T = \frac{zZe^2}{4\pi\epsilon_0 D}$ expression relating kinetic energy and the closest approach for head-on collision, one finds

$$F \;= \frac{TD}{r^2}$$

. The component of this force in the direction of $\mathbf{q}$ is

$$F_{\mathbf{q}}(t) \; = \frac{TD}{r^2} \cos\psi(t)$$

and, therefore, the change of momentum ($F_{\mathbf{q}}(t) = \frac{d\mathbf{q}}{dt}$) is given by

$$q \;=\; \int \frac{zZe^2}{4\pi\epsilon_0 r^2} \cos\psi \, dt. \tag{2.1.3}$$

We can replace integration over time by integration over the angle ψ using

$$dt \;=\; \frac{d\psi}{\dot{\psi}},$$

where $\dot{\psi}$ can be obtained form conservation of angular momentum,

$$L \;=\; m_\alpha r^2 \dot{\psi}.$$

The initial angular momentum is given by

$$L \;=\; bp,$$

so we have

$$\dot{\psi} \;=\; \frac{b\,p}{m_\alpha\, r^2},$$

so that eq.(2.1.3) becomes

$$q \;=\; \int \frac{TD\, m_\alpha\, r^2}{r^2\, b\, p} \cos\psi\, d\psi \;=\; \int \frac{D\, p}{2b} \cos\psi\, d\psi, \tag{2.1.4}$$

where kinetic energy of α-particle $T = p^2/(2m_\alpha)$ related its momenta and its mass was substituted at the last step. Note that r^2 has cancelled.

From the diagram we see that the limits on ψ are

$$\psi \;=\; \pm\frac{1}{2}(\pi - \theta),$$

so that we get

$$q \;=\; \frac{Dp}{2b} 2sin\left(\frac{1}{2}(\pi - \theta)\right)$$

Now using eq.(2.1.2) we get

$$2p\sin\left(\frac{\theta}{2}\right) \;=\; \frac{Dp}{2b} 2sin\left(\frac{1}{2}(\pi - \theta)\right)$$

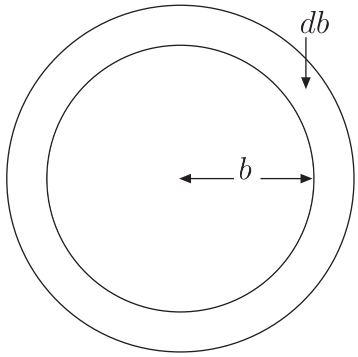
from where it follows that

$$\tan(\theta/2) = \frac{D}{2b}$$

2.2 Flux and cross-section

The "flux", F of incident particles is defined as the number of incident particles arriving per unit area per second at the target.

The number of particles, $dN(b)$, with impact parameter between b and $b+db$ is this flux multiplied by the area between two concentric circles of radius b and $b+db$



$$dN(b) \;=\; F\,2\pi b\,db \tag{2.2.5}$$

Differentiating eq.(2.1.1) gives us

$$db = -\frac{D}{4\sin^2(\theta/2)}d\theta \tag{2.2.6}$$

which allows us to write an expression for the number of α-particles scattered through an angle between θ and $\theta + d\theta$ after substitution Eq.(2.2.6) and Eq.(2.1.1) into Eq.(2.2.5):

$$dN(\theta) \;= F\pi\frac{D^2}{4}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}d\theta. \tag{2.2.7}$$

(the minus sign has been dropped as it merely indicates that as b increases, the scattering angle θ decreases - $N(\theta)$ must be positive).

The "differential cross-section", $d\sigma/d\theta$, with respect to the scattering angle is the number of scatterings between θ and $\theta + d\theta$ per unit flux, per unit range of angle, i.e.

$$\frac{d\sigma}{d\theta} \;=\frac{dN(\theta)}{Fd\theta} \;=\; \pi\frac{D^2}{4}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}.$$

It is more usual to quote the differential cross-section with respect to a given solid angle Ω, which is related to the scattering angle θ and the azimuthal angle ϕ by

$$d\Omega \;=\; \sin\theta d\theta d\phi \;=\; 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta d\phi.$$

The relation between the number of events, the flux, differential solid angle $d\Omega$ and differential cross section is given by

$$\frac{dN}{d\Omega} = F\frac{d\sigma}{d\Omega}$$

. in analogy to the relation for differential angle $d\theta$.

The integrastion over the azimuthal angle just gives a factor of 2π so we may write

$$\frac{d\sigma}{d\theta} \;=\; 2\pi\frac{d^2\sigma}{d\theta d\phi}$$

so that

$$\frac{d^2\sigma}{d\theta d\phi} \;=\; \frac{D^2}{8}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}.$$

and substitute $d\theta d\phi$ by $d\Omega$ (using the above relation) to obtain

$$\frac{d\sigma}{d\Omega} \;=\; \frac{D^2}{8}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}\frac{1}{2\sin(\theta/2)\cos(\theta/2)} \;=\; \frac{D^2}{16\sin^4(\theta/2)}.$$

Differential cross-sections have the dimension of an area. These are usually quoted in terms of "barns". I barn is defined to be $10^{-28}\,m^2$, so that, for example, 1 millibarn (mb) is an area of $10^{-31}\,m^2$.

The unit of length that is often used in nuclear physics is the "fermi" (fm) which is defined to be 10^{-15} m and energies are usually quoted in electron volts (Kev, MeV, or GeV). A cross-section of 1 fm^2 corresponds ot 10 mb. For the purposes of numerical calculations, it is worth noting that

$$\hbar c \;=\; 197.3\,\mathrm{MeV\,fm},$$

so that

$$\frac{e^2}{4\pi\epsilon_0} \;=\; \alpha\,\hbar c \;=\; \frac{1}{137}\times 197.3\,\mathrm{MeV\,fm}$$

For example, the distance of closest approach is therefore given by

$$D \;=\; \frac{197.3}{137}\frac{zZ}{T}\,\mathrm{fm},$$

where the kinetic energy T is given in MeV.

2.3 Results and interpretation of the Rutherford experiment

Although the differential cross-section falls rapidly with the scattering angle, the cross-section at large angles is still much larger than would have been obtained from Thomson's 'current cake' model of the atom in which electrons are embedded in a 'dough' of positive charge -

so that as the α-particle moves through the atom it suffers a large number of small-angle scatterings in random directions.

We notice that the differential cross-section diverges as the scattering angle goes to zero. However we note from eq.(2.1.1) that small angle scattering implies a large impact parameter. The distance of the incident particle from any nucleus can only grow to about half of the distance between the nuclei in the gold foil. In fact, the total number of particles scattered into a given solid angle is the differential cross-section multiplied by the flux, multiplied by the number of nuclei in the foil - or more precisely in the part of the foil that is 'illuminated' by the incident α-particles. We assume that the foil is sufficiently thin so that multiple scatterings are very unlikely and we can make the approximation that all the nuclei lie in a single plane. The mass of a nucleus with atomic mass number A is given to a very good approximation by Am_p, total number of nuclei per unit area of foil is given by

$$\rho\delta\frac{1}{Am_p}$$

where ρ is the density, δ is the thickness of the foil, A is the atomic mass. This means that the fraction of α particles scattered into a small interval of solid angle $d\Omega$ is given by

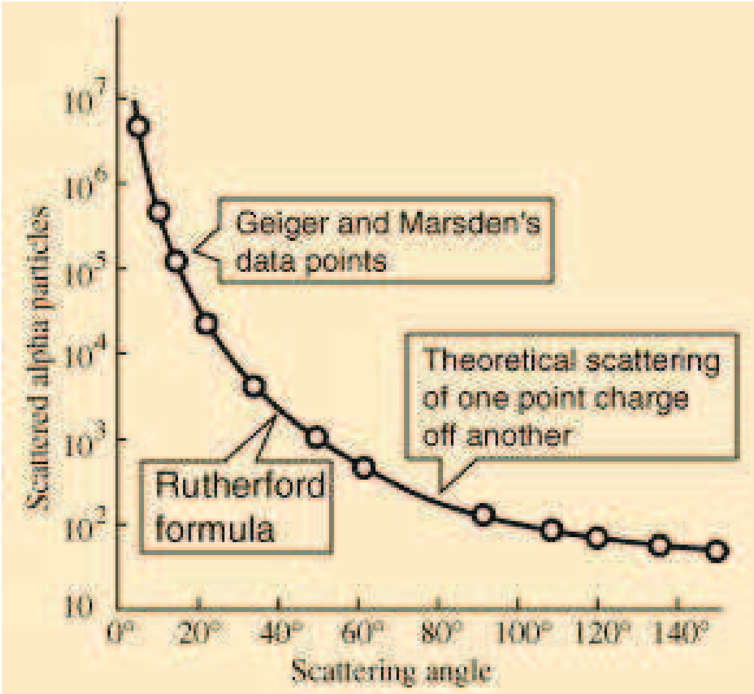
$$\frac{\delta n}{n} = \rho\delta\frac{1}{Am_p}\frac{d\sigma}{d\Omega}d\Omega \tag{2.3.8}$$

Solid angle is defined such that an area element dA at a distance r from the scattering centre subtends a solid angle

$$d\Omega = \frac{dA}{r^2},$$

so that if we place a detector with an acceptance area dA at a distance r from the foil and at an angle θ to the direction of the incident α-particles then the fraction of incident α-particles enter the detector is given by replacing $d\Omega$ by dA/r^2 in eq.(2.3.8)

This theoretical result compares very well with the data taken by Geiger and Marsden.


Geiger and Marsden's data points
Theoretical scattering of one point charge off another
Rutherford formula
Scattered alpha particles
10^7
10^6
10^5
10^4
10^3
10^2
10
0° 20° 40° 60° 80° 100° 120° 140°
Scattering angle